An Improvement on RPA Based on a Boson description

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Abstract

We use a solvable model to perform modified Dyson mapping and reveal the unphysical-state effects in the original Random Phase Approximation (RPA). We then propose a method to introduce the RPA and improve it based on the Boson description.

I.Introduction

The RPA is often used to describe collective excitations. In nuclear physics it is widely employed to define surface oscillation and pair oscillation[1]. Recently the RPA has also been used to study mesons in terms of the quark-antiquark degrees of freedom[2, 3].

The RPA builds bosonic excitations with fermions so it includes unphysical state effects. In view of its important role in practice, it is helpful to explore convenient approaches for improvement. In this paper we will develop such a method by studying a simple solvable model with the help of the modified Dyson mapping [4, 5]. We will introduce the RPA in a different way and explain it as the zero order approximation of a general Boson description. Then only very low orders of the correction terms will be included so as to preserve the simplicity of the method. It will be seen that our method are both quite powerful. Although the model is very simple our method can be, in principle, generalized approximately.

II. Model and Boson Description for Particle-Antiparticle Pair States

Consider a simple system of identical Fermions and their anti-particles and suppose that each particle has only two independent states. The Hamiltonian is assumed to be:

$$H = \epsilon \sum_{\mu = \pm 1} (b_{\mu}^{\dagger} b_{\mu} + d_{\mu}^{\dagger} d_{\mu}) + R(b_{1}^{\dagger} d_{-1}^{\dagger} b_{-1}^{\dagger} d_{1}^{\dagger} + d_{1} b_{-1} d_{-1} b_{1}) ,$$

where $b^{\dagger}_{\mu}|0\rangle$ and $d^{\dagger}_{\mu}|0\rangle$ stand for the particle states and anti-particle states respectively. The state vector of the ground state of the system is

$$|\Psi_0\rangle = x_1|0\rangle + x_2b_1^{\dagger}d_{-1}^{\dagger}b_{-1}^{\dagger}d_1^{\dagger}|0\rangle,$$

with

$$\frac{x_2}{x_1} = \frac{E_0}{R} = \frac{2\epsilon}{R} - \sqrt{(\frac{2\epsilon}{R})^2 + 1} \,,$$

where E_0 is the ground state energy:

$$E_0 = 2\epsilon - \sqrt{(2\epsilon)^2 + R^2}.$$

The particle-antiparticle pair states $b_1^{\dagger}d_{-1}^{\dagger}|0\rangle$ and $b_{-1}^{\dagger}d_1^{\dagger}|0\rangle$ are also the exact eigen states of H belonging to the eigenvalue 2ϵ . Namely,

$$\left\{ \begin{array}{l} Hb_1^\dagger d_{-1}^\dagger |0\rangle = 2\epsilon b_1^\dagger d_{-1}^\dagger |0\rangle \,, \\ Hb_{-1}^\dagger d_1^\dagger |0\rangle = 2\epsilon b_{-1}^\dagger d_1^\dagger |0\rangle \,. \end{array} \right.$$

On the other hand the RPA excitation of a particle-antiparticle pair takes the following form

$$Q_1^{\dagger} = X b_1^{\dagger} d_{-1}^{\dagger} + Y d_1 b_{-1} \,,$$

$$Q_{-1}^{\dagger} = X d_1^{\dagger} b_{-1}^{\dagger} + Y b_1 d_{-1} ,$$

where X,Y are real numbers and to be found from the RPA equation:

$$\begin{cases} [H, Q_{\mu}^{\dagger}] \approx \omega_{RPA} Q_{\mu}^{\dagger} \\ [Q_{\mu}, Q_{\nu}^{\dagger}] \approx \delta_{\mu\nu} \end{cases}$$

The solution is:

$$\left\{ \begin{array}{l} \omega_{RAP} = \sqrt{(2\epsilon)^2 - R^2} \\ \frac{Y}{X} = \frac{2\epsilon}{R} - \sqrt{(\frac{2\epsilon}{R})^2 - 1} \end{array}, (X^2 - Y^2 = 1) \right. .$$

 ω_{RAP} is the excited energy with respect to the RPA ground state which is determined by the condition:

$$Q_{\mu}|\Psi_{0}^{RPA}\rangle=0$$
.

One gets from this

$$|\Psi_0^{RPA}\rangle = X_1|0\rangle + X_2b_1^{\dagger}d_{-1}^{\dagger}b_{-1}^{\dagger}d_1^{\dagger}|0\rangle \,,$$

with

$$\frac{X_2}{X_1} = -\frac{Y}{X} = -\frac{2\epsilon}{R} + \sqrt{(\frac{2\epsilon}{R})^2 - 1} \ (X_1^2 + X_2^2 = 1).$$

It can be seen clearly that the RPA worsens when $\frac{2\epsilon}{R} \to 1$ and breaks down when $\frac{2\epsilon}{R} < 1$. Since the RPA operators Q_{μ} , Q_{μ}^{\dagger} are assigned to satisfy the standard Boson commutation rule, it should be convenient to analyse the RPA within an exact Boson description. For this purpose, it is enough to consider the Boson image of the invariant subspace V_F of H, which is spanned by the states:

$$|0\rangle\,,b_1^\dagger d_{-1}^\dagger |0\rangle\,,b_1^\dagger d_1^\dagger |0\rangle\,,b_1^\dagger d_{-1}^\dagger b_1^\dagger d_1^\dagger |0\rangle\,.$$

We now employ the modified Dyson mapping to set up a 1-1 correspondence between the fermion state subspace V_F and a Boson state subspace V_B . The latter is spanned by the four Boson states

$$|0)\,,\ B_1^\dagger|0)\,,\ B_{-1}^\dagger|0)\,,\ B_1^\dagger B_{-1}^\dagger|0)\,,$$

where B_{μ}^{\dagger} are Boson creation operators, and B_{μ} are the corresponding annihilation operators. They of course satisfy the standard Boson commutation rule. These four Boson states are to be described as the Boson images of the above four Fermion states. Therefore, $|0\rangle$ has the same quantum numbers as $|0\rangle$ and satisfies

$$B_{\mu}|0\rangle = 0$$
, $(0|0\rangle = 1$,

and $B_1^\dagger|0\rangle$ and $B_{-1}^\dagger|0\rangle$ carry the quantum numbers of $b_1^\dagger d_{-1}^\dagger|0\rangle$ and $b_{-1}^\dagger d_1^\dagger|0\rangle$ respectively. The correspondence can be expressed as

$$\Phi = \Gamma_B U \Psi \,,$$

$$\Psi = U^{\dagger} \Gamma_B \Phi .$$

where

$$U = \langle 0|e^{B_1^{\dagger}d_{-1}b_1 + B_{-1}^{\dagger}d_1b_{-1}}|0\rangle,$$

$$U^{\dagger} = (0|e^{b_1^{\dagger}d_{-1}^{\dagger}B_1 + b_{-1}^{\dagger}d_1^{\dagger}B_{-1}}|0\rangle.$$

 Γ_B is the projection operator to the subspace V_B . Namely:

$$\Gamma_B \Phi = \left\{ \begin{array}{l} \Phi \text{ when } \Phi \in V_B \\ 0 \text{ when } \Phi \perp V_B. \end{array} \right.$$

According to this correspondence, one can easily find the Boson image of the Hamiltonian and write it as

$$H_B = \Gamma_B \{ 2\epsilon \sum_{\mu} B_{\mu}^{\dagger} B_{\mu} + R(B_1^{\dagger} B_{-1}^{\dagger} + B_{-1} B_1) \} \Gamma_B .$$

This Boson Hamiltonian and the V_B construct an exact Boson description for the particle-antiparticle pair states. Each Boson state of the subspace V_B is a physical one, and a Boson state with components outside V_B is partially or totally unphysical. In this description, the 1-Boson eigenstates of H_B are $B^{\dagger}_{\mu}|0\rangle$ which correspond to the 1-pair Fermion states, and the ground state is:

$$|\Phi_0\rangle = x_1|0\rangle + x_2B_1^{\dagger}B_{-1}^{\dagger}|0\rangle$$
.

III. An Improvement on RPA Based on the Boson description

The excitation elements of RPA are expressed in terms of Fermion operators and assumed to satisfy the standard Boson commutation rule. This cause some unphysical state effects which can clearly be seen from a Boson description. The exact Boson Hamiltonian H_B can be written as

$$H_B = \Gamma_B(\omega_0 + \omega_{RPA} \sum_{\mu} A^{\dagger}_{\mu} A_{\mu}) \Gamma_B = \Gamma_B H_A \Gamma_B ,$$

where

$$\begin{split} \omega_0 &= -2\epsilon + \sqrt{(2\epsilon)^2 - R^2} \,, \\ A_\mu^\dagger &= X B_\mu^\dagger + Y B_{-\mu} \,, \\ H_A &= \omega_0 + \omega_{RPA} \sum_\mu A_\mu^\dagger A_\mu \,. \end{split}$$

If we treat the operator H_A as a true Hamiltonian and solve its eigen equation in the whole Boson state space than we get the RPA results. Such a kind of eigenstates of H_A of course contain many unphysical components. For a further study we now rely on the Boson description to introduce the RPA in a new point of view. Namely, we regard the following as the Boson Hamiltonian of RPA:

$$H_B^{RPA} = \Gamma_B(\omega_0 + \omega_{RPA} \sum_{\mu} B_{\mu}^{\dagger} B_{\mu}) \Gamma_B .$$

Next, we will explain the RPA as the zeroth order approximation of our exact Boson description. For this purpose we make the transformation :

$$\begin{cases} \tilde{B}^{\dagger}_{\mu} = \beta B^{\dagger}_{\mu} + \alpha B_{-\mu} & (\beta^2 - \alpha^2 = 1), \\ B^{\dagger}_{\mu} = \beta \tilde{B}^{\dagger}_{\mu} - \alpha \tilde{B}_{-\mu} \end{cases}$$

and write H_B as

$$H_{B} = \Gamma_{B} \{ E'_{0} + \omega' \sum_{\mu} \tilde{B}^{\dagger}_{\mu} \tilde{B}_{\mu} + R' (\tilde{B}^{\dagger}_{1} \tilde{B}^{\dagger}_{-1} + \tilde{B}_{-1} \tilde{B}_{1}) \} \Gamma_{B} ,$$

with

$$E'_0 = 4\epsilon\alpha^2 - 2\alpha\beta R,$$

$$\omega' = 2\epsilon(\alpha^2 + \beta^2) - 2\alpha\beta R,$$

$$R' = (\alpha^2 + \beta^2)R - 4\alpha\beta\epsilon.$$

Introducing S and γ

$$S = e^{-\gamma (B_1^{\dagger} B_{-1}^{\dagger} - B_{-1} B_1)},$$

$$e^{\gamma} = \alpha + \beta,$$

$$e^{-\gamma} = \beta - \alpha,$$

one has

$$\begin{split} \tilde{B}_{\mu}^{\dagger} &= S B_{\mu}^{\dagger} S^{\dagger} \,, \\ H_B &= \Gamma_B S \{ E_0^{\prime} + \omega^{\prime} \sum_{\mu} B_{\mu}^{\dagger} B_{\mu} + R^{\prime} (B_1^{\dagger} B_{-1}^{\dagger} + B_{-1} B_1) \} S^{\dagger} \Gamma_B \,. \end{split}$$

If α and β are set to be X and Y respectively then one has

$$H_B = \Gamma_B S_{RPA} (\omega_0 + \omega_{RPA} \sum_{\mu} B_{\mu}^{\dagger} B_{\mu}) S_{RPA}^{\dagger} \Gamma_B .$$

In other word, the RPA Hamiltonan H_B^{RPA} can be found from the expression for H_B by replacing the operator S with 1 and then determining α and β with the help of the condition R' = 0. Our method to improve the RPA is to expand S in powers of γ and keep only a few terms to get an approximate Hamiltonian:

$$H_B \approx \Gamma_B \{ \overline{E_0} + \overline{\omega} \sum_{\mu} B_{\mu}^{\dagger} B_{\mu} + \overline{R} (B_1^{\dagger} B_{-1}^{\dagger} + B_{-1} B_1) \} \Gamma_B .$$

Then the parameters are determined by the condition $\overline{R} = 0$, and the following formulae:

$$\overline{E_0} = E'_0 + 2\gamma R' + 2\gamma^2 \omega' + \cdots,$$

$$\overline{\omega} = \omega' + 2\gamma R' + 2\gamma^2 \omega' + \cdots,$$

$$\overline{R} = R' + 2\gamma \omega' + 2\gamma^2 R' + \cdots.$$

We expect that our simple scheme (MRPA) will be useful in improving the behaviour of the PRA, especially in the region where the RPA begins to break down. Numerical results have confirmed our expectation. We present here results to second order of γ . We have also calculated the excitation energy to the fourth order and found it is a little better. This shows our method is stable and self-consistent.

The figure is the one Boson excitation energy versus $\frac{2\epsilon}{R}$. The energy unit is R. In order to show more clearly the improvement when $\frac{2\epsilon}{R} \to 1$, the abscissa is assumed to stand for $\frac{2\epsilon}{R} - 1$ and is scaled logrithmically. When $\frac{2\epsilon}{R} \to \infty$, the results of the RPA and our method are close to the exact solution. When $\frac{2\epsilon}{R} = 1$, both the RPA and our method give zero excitation energy. In the region where $\frac{2\epsilon}{R}$ is larger than 1 but not too large, the one Boson excitation energy of RPA deviates from the exact solution significantly and approaches zero rapidly while our method gives a much better description. The trend our method approaches zero is very slow which is an important improvement.

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References

- [1] Aage Bohr and Ben R. Mottleson , Nuclear structure **Vol. II** Nuclear deformation (W. A. Benjamin , Inc. 1975)
- [2] V. Bernard , R. Brockmann and W. Weise , Nucl. Phys. **A440**(1985)605
- [3] W. Z. Deng J. W. Zhang and L. M. Yang Commun. Theor. Phys. 21(1994)311
- [4] Z.-S. Yang , Liu Yong et al , Nucl. Phys. A421(1984)297
- [5] Z.-S. Yang , H. Qi , X.-H. Li , A. Arima and T. Otsuka , A possible way to generalize the O-A-I method to non-degenerate j shells, in : Symmetries in Science VII , ed. Bruno Gruber and Takaharu Otsuka (Plenum Publishing Corporation , 1994) PP.595–604